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2008 J. Phys. A: Math. Theor. 41 255203

(http://iopscience.iop.org/1751-8121/41/25/255203)

J. Phys. A: Math. Theor. **41** (2008) 255203 (5pp) doi:10.1088/1751-8113/41/25/255203

High-frequency limit of the transport cross section in scattering by an obstacle with impedance boundary conditions

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Received 26 February 2008, in final form 7 May 2008
Published 28 May 2008
Online at stacks.iop.org/JPhysA/41/255203

Abstract
The scalar scattering of a plane wave by a strictly convex obstacle with impedance boundary conditions is considered. A uniform bound of the total cross section for all values of the frequency is presented. The high-frequency limit of the transport cross section is calculated and presented as a classical functional of the variational calculus.

PACS numbers: 43.20.−f, 11.55.-m,

1. Introduction

Consider a strictly convex body $\Omega \subset \mathbb{R}^3$ with C^2-boundary $\partial\Omega$ and $k > 0$. The scattered field is given by the Helmholtz equation and a radiation condition

$$\Delta u(r) + k^2 u(r) = 0, \qquad r \in \Omega' = \mathbb{R}^3 \backslash \Omega \tag{1}$$

$$\int_{|r|=R} \left| \frac{\partial u(r)}{\partial |r|} - \mathrm{i}ku(r) \right|^2 \mathrm{d}S = o(1), \qquad R \to \infty, \tag{2}$$

with Dirichlet, Neumann or impedance boundary conditions of the form

$$\mathcal{B}_\gamma(u)|_{\partial\Omega} \equiv -\mathcal{B}_\gamma(\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)})|_{\partial\Omega}, \qquad r = (x, y, z) \in \partial\Omega, \tag{3}$$

where $\gamma > 0$ is a constant, $\mathcal{B}_\gamma = (\partial/\partial n) + \mathrm{i}k\gamma$ and $\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}$ is an incident field formed by a plane wave with incident angle $\theta_0 = (0, 0, 1) \in S^2$. The operator B_γ appears (for example [1]) as a stationary analog of the $\frac{\partial}{\partial n} - \gamma\frac{\partial}{\partial t}$ for the non-reduced wave equation. In [2, 3] the existence and uniqueness of the solution of (1)–(3) is proven. A function $u(r)$ which satisfies the mentioned conditions has the asymptotic

$$u(r) = \frac{\mathrm{e}^{\mathrm{i}k|r|}}{|r|} f_\gamma(\theta) + o\left(\frac{1}{|r|}\right), \qquad r \to \infty, \qquad \theta = r/|r| \in S^2, \tag{4}$$

where the function $f_\gamma(\theta) = f_\gamma(\theta, k)$ is called *scattering amplitude* and the quantity

$$\sigma_\gamma = \int_{S^2} |f_\gamma(\theta)|^2 \, \mathrm{d}\mu(\theta)$$

is called the total cross section. μ is a square element of the unit sphere. The projection on the incident direction θ_0 of the total momentum transmitted to the obstacle is given by a quantity called transport cross section (for a large volume normalization)

$$R_\gamma = \int_{S^2} \langle \theta_0 - \theta, \theta_0 \rangle |f_\gamma(\theta)|^2 \, \mathrm{d}\mu(\theta). \tag{5}$$

The case of impedance boundary conditions (i.e. finite values of $\gamma > 0$) is not completely studied. We know [1, Theorem 1] that uniformly on every open subset of $\{\theta \in S^2 : \theta \neq \theta_0\}$

$$f_\gamma(\theta) = \frac{1}{2}\mathcal{K}(y^+)^{-1/2} \, \mathrm{e}^{\mathrm{i}k\langle y^+(\theta)\cdot(\theta-\theta_0)\rangle} \left(\frac{\gamma - \langle \boldsymbol{n}(\theta), \theta \rangle}{\gamma + \langle \boldsymbol{n}(\theta), \theta \rangle} \right) + O(1/k), \qquad k \to \infty. \tag{6}$$

Here $y^+(\theta) \in \partial\Omega$ is the preimage of $\boldsymbol{n}(\theta) := (\theta - \theta_0)/(|\theta - \theta_0|) \in S^2$ under the Gauss map (for example [4]), $\mathcal{K}(y^+)$ is the Gauss curvature at $y^+ \in \partial\Omega$. But unfortunately the behavior of σ_γ for $\gamma > 0$ and large values of k is not known since we do not know the behavior of the scattering amplitude near the forward direction θ_0 and therefore we cannot calculate limits of observables like R_γ i.e. even if integrand vanishes at $\theta_0 = 0$. But such calculations become possible if we prove that σ_γ is bounded from infinity uniformly for all (large enough) values of k. In other words, we should prove that the contribution of the diffraction peak to the total cross section is uniformly bounded. Note that this fact is known for the cases of Dirichlet or Neumann boundary conditions. We will discuss it in part 3.

Theorem 1.

(1) *The following inequality holds for all values of* $k > 0$*:*

$$\sigma_\gamma \leqslant 2S(\partial\Omega)\frac{(1+\gamma)^2}{\gamma}, \tag{7}$$

where $S(\partial\Omega)$ *is the area of the* $\partial\Omega$.

(2) *Let the visible part of* Ω *be written as a graph of the smooth function* $g(x) : \mathcal{I} \to \mathbb{R}^3$*, where* $\mathcal{I} \in \mathbb{R}^2$ *is the part of the plane perpendicular to* θ_0*; then*

$$\lim_{k\to\infty} R_\gamma = \int_{\mathcal{I}} \frac{2\,\mathrm{d}x}{1+|\nabla g|^2} \left(\frac{\gamma\sqrt{1+|\nabla g|^2} - 1}{\gamma\sqrt{1+|\nabla g|^2} + 1} \right)^2. \tag{8}$$

The cases $\gamma = 0$ and $\gamma = \infty$ correspond to the classical resistance functional which was investigated starting from Newton [5, 1685] and in many recent articles (for example [6–8]). For the cases $0 < \gamma < \infty$, note that representation (8) has the form of a standard functional and it provides a means for further study using Optimization Theory.

2. Proofs

Let u be the field of the outgoing wave satisfying (1), (2), (3). Everywhere below $\|\cdot\| = \|\cdot\|_{L_2(\partial\Omega,\mathrm{d}S)}$. Let us prove that

$$\left\| \frac{\partial u}{\partial n} + \mathrm{i}k\gamma u \right\| \geqslant k\gamma \|u\|. \tag{9}$$

Start noting that

$$\left\|\frac{\partial u}{\partial n}+\mathrm{i}k\gamma u\right\|^2=\left\|\frac{\partial u}{\partial n}\right\|^2+2k\gamma\,\mathrm{Im}\left(\int_{\partial\Omega}\frac{\partial u}{\partial n}\overline{u}\,\mathrm{d}S\right)+(k\gamma)^2\|u\|^2.$$

Using the well-known fact (which follows from the Second Green's identity)

$$\mathrm{Im}\left(\int_{\partial\Omega}\frac{\partial u}{\partial n}\overline{u}\,\mathrm{d}S\right)=k\sigma_\gamma\geqslant 0, \tag{10}$$

we obtain (9). Note now that from (3), it follows that

$$\begin{aligned}\left\|\frac{\partial u}{\partial n}+\mathrm{i}k\gamma u\right\|&=\left\|\frac{\partial\,\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}}{\partial n}+\mathrm{i}k\gamma\,\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}\right\|\\&\leqslant\left\|\frac{\partial\,\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}}{\partial n}\right\|+k\gamma\|\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}\|\leqslant\sqrt{S}k(1+\gamma).\end{aligned}$$

Here and below $S=S(\partial\Omega)$. So using (9), we obtain

$$\gamma\|u\|\leqslant\sqrt{S}(1+\gamma). \tag{11}$$

Also from (3) we have

$$-\frac{\partial u}{\partial n}=\mathrm{i}k\gamma u+\frac{\partial\,\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}}{\partial n}+\mathrm{i}k\gamma\,\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)},$$

therefore

$$\begin{aligned}\left\|\frac{\partial u}{\partial n}\right\|&\leqslant k\gamma\|u|+\left\|\frac{\partial\,\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}}{\partial n}\right\|+k\gamma\|\mathrm{e}^{\mathrm{i}k(r\cdot\theta_0)}\|\\&\leqslant k\gamma\|u\|+\sqrt{S}k(1+\gamma)\leqslant 2k\sqrt{S}(1+\gamma).\end{aligned}$$

Now from (10) and (11) we have

$$\sigma_\gamma\leqslant\frac{1}{k}\|u\|\left\|\frac{\partial u}{\partial n}\right\|\leqslant\frac{1}{k}\left(\frac{\sqrt{S}(1+\gamma)}{\gamma}\right)(2k\sqrt{S}(1+\gamma))=\frac{2S(1+\gamma)^2}{\gamma}. \tag{12}$$

This ends the proof of the first part of theorem 1.

From (6) we have for $k\to\infty$

$$|f_\gamma(\theta)|^2=\frac{1}{4}\mathcal{K}(y^+(\theta))^{-1}\left(\frac{\gamma-\langle\boldsymbol{n}(\theta),\theta\rangle}{\gamma+\langle\boldsymbol{n}(\theta),\theta\rangle}\right)^2+O(1/k),\qquad\theta\neq\theta_0, \tag{13}$$

where estimate $O(1/k)$ in (13) is uniform on compact subsets of $\{\theta\in S^2|\theta\neq\theta_0\}$.

Using (7) and the fact that the density of the integral (5) is continuous and it becomes zero in the point $\theta=\theta_0$, we obtain for $k\to\infty$

$$\begin{aligned}R_\gamma&=\int_{S^2}(1-\langle\theta,\theta_0\rangle)|f_\gamma(\theta)|^2\,\mathrm{d}\mu(\theta)\\&=\int_{S^2}(1-\langle\theta,\theta_0\rangle)(4\mathcal{K}(y^+(\theta)))^{-1}\left(\frac{\gamma-\langle\boldsymbol{n}(\theta),\theta\rangle}{\gamma+\langle\boldsymbol{n}(\theta),\theta\rangle}\right)^2\mathrm{d}\mu(\theta)+o(1).\end{aligned} \tag{14}$$

We now define a change of variables $\theta(x):\mathcal{I}\to S^2$. Let $\boldsymbol{n}(x)$ be an outward normal in the point $y^+(x)=(x,g(x))\in\partial\Omega$, then we put $\theta(x)=\theta_0-2\langle\boldsymbol{n}(x),\theta_0\rangle\boldsymbol{n}$. It is easy to see that $\theta(x)\in S^2$ and this map is one-to-one, since the obstacle Ω is strictly convex and therefore the Gauss map $\boldsymbol{n}(x)$ is also one-to-one.

Let us introduce standard spherical coordinates $(\cos(\widetilde{\theta}), \widetilde{\varphi}) \in [-1, 1] \times [0, 2\pi)$. We now calculate the Jacobian $D(\cos(\widetilde{\theta}), \widetilde{\varphi})/D(x_1, x_2)$, where $(x_1, x_2) = x$ are orthonormal coordinates on $\mathcal{I}$.

Note that

$$\theta(x) = \left(\frac{2g'_{x_1}}{1+|\nabla g|^2}, \frac{2g'_{x_2}}{1+|\nabla g|^2}, \frac{|\nabla g|^2-1}{|\nabla g|^2+1}\right)$$

$$\cos(\widetilde{\theta}) = 1 - \frac{2}{|\nabla g|^2+1},$$

$$\widetilde{\varphi} = \arctan(g'_{x_2}/g'_{x_1})$$

$$\begin{aligned}\frac{D(\cos(\widetilde{\theta}), \widetilde{\varphi})}{D(x_1, x_2)} &= \begin{vmatrix}\cos(\widetilde{\theta})'_{x_1} \widetilde{\varphi}'_{x_1} \\ \cos(\widetilde{\theta})'_{x_2} \widetilde{\varphi}'_{x_2}\end{vmatrix} \\ &= \frac{4}{(1+|\nabla g|^2)(|\nabla g|^2)} \begin{vmatrix} g'_{x_1} g''_{x_1x_1} + g'_{x_2} g''_{x_1x_2} g'_{x_1} g''_{x_1x_2} - g'_{x_2} g''_{x_1x_1} \\ g'_{x_1} g''_{x_1x_2} + g'_{x_2} g''_{x_2x_2} g'_{x_1} g''_{x_2x_2} - g'_{x_2} g''_{x_1x_2}\end{vmatrix} \\ &= \frac{4((g''_{x_1x_2})^2 - g''_{x_1x_1} g''_{x_2x_2})}{1+|\nabla g|^2} = -4\mathcal{K}(x_1, x_2). \end{aligned} \tag{15}$$

Note now that for $x \in \mathcal{I}$

$$\langle\theta_0 - \theta(x), \theta_0\rangle = \langle 2\langle \boldsymbol{n}(x), \theta_0\rangle \boldsymbol{n}, \theta_0\rangle = 2\langle \boldsymbol{n}(x), \theta_0\rangle^2 = \frac{2}{1+|\nabla u(x)|^2}. \tag{16}$$

Applying (16) and (15) for (14) we obtain (8). Theorem 1 is proved.

3. Discussions

As we mentioned in the beginning of the paper, the uniform boundedness of the total cross section gives us the possibility of calculating the high-frequency limit of observables with integrand vanishing at $\theta_0 = 0$. Investigation of other observables could be done after description of scattering amplitude as it is done in Dirichlet or Neumann cases:

In the case of Dirichlet (or Neumann) boundary conditions, the limit behavior of the scattering amplitude in the high-frequency regime has been described completely by the following two statements [1, theorem 1]:

$$f_\infty(\theta) = \tfrac{1}{2}\mathcal{K}(y^+(\theta))^{-1/2}\, \mathrm{e}^{\mathrm{i}k\langle y^+(\theta)\cdot(\theta-\theta_0)\rangle} + O(1/k), \qquad \theta \neq \theta_0. \tag{17}$$

The estimate $O(1/k)$ is uniform on compact subsets of $\{\theta \in S^2 | \theta \neq \theta_0\}$.

The behavior near the forward directions is given by (see [9, 10])

$$\lim_{k\to\infty} |f_\infty|^2 = |f|^2_{\mathrm{cl}} + \sigma_{\mathrm{cl}}\delta(\theta_0), \tag{18}$$

where $|f_{\mathrm{cl}}(\theta)|^2 = (2K(y^+(\theta)))^{-1}$ is the classical density of scattered rays, $\delta(\cdot)$ is a Dirac delta function, and the limit is in the sense of distributions. This formula allows one to obtain limits at high k of all quantities such as

$$\int_{S^2} \varphi(\theta)|f_\infty(\theta)|^2 \mathrm{d}\mu(\theta), \qquad \varphi \in C(S^2). \tag{19}$$

In particular, we have

$$\lim_{k\to\infty} \sigma_\infty = 2\sigma_{\mathrm{cl}}, \qquad \lim_{k\to\infty} R_\infty = R_{\mathrm{cl}}. \tag{20}$$

Here σ_{cl}, R_{cl} are the classical total cross section and classical transport cross section. The limit of σ_∞ in the case of a sphere is calculated in many physics textbooks, and for the case of convex bodies there is a rigorous proof in [11]. Moreover, this fact ($\lim_{k\to\infty} \sigma_\infty = 2\sigma_{cl}$) was used in [9] to prove (18).[1]

Acknowledgments

This work was supported by the Centre for Research on Optimization and Control (CEOC) from the '*Fundação para a Ciência e a Tecnologia*' (FCT), cofinanced by the European Community Fund FEDER/POCTI, and by the FCT research project PTDC/MAT/72840/2006.

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[1] See [11] for proof of $\lim_{k\to\infty} \sigma_0 = 2\sigma_{cl}$.